Normal and anomalous scaling in a problem of a passively advected magnetic field

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Recently M. Vergassola [Phys. Rev. E **53**, R3021 (1996)] considered the possibility of anomalous scaling in the three-dimensional dynamo problem. It has been shown that the two-point correlation function of magnetic field, advected by a white-in-time random velocity field with zero helicity has anomalous inertial-range scaling exponent in a statistically steady state. In this work we demonstrate that for the same problem the scaling of covariance of magnetic helicity is normal. The difference in scalings comes from the fact that unlike magnetic energy magnetic helicity in this problem is conserved. It is also conjectured that even small violation of parity invariance of the velocity field makes existence of the steady-state solution impossible. [S1063-651X(96)51506-0]

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To quantitatively describe hydrodynamic turbulence or turbulent transport phenomena one has to solve partial differential equation for dynamical variables (local velocity of a fluid, concentration of a contaminant, etc.) advected by a random (turbulent) velocity field. The Navier-Stokes and a passive scalar equations are the most prominent examples. These equations introduce inviscid dynamic constraints, such as energy and scalar variance conservation laws, which play a very important part in turbulence theory. If an external energy input comes from the largest scales in a system and a viscous (diffusive) dissipation acts at the small scales only we, assuming the existence of a statistically steady state in the limit of zero viscosity (diffusivity), come to the only conclusion that the dynamics of the intermediate scales are dominated by the advective (nonlinear) contributions to the equation. This range of scales is called inertial range. In a statistically steady state due to the conservation laws, the dynamics of fluctuations in the inertial range of scales can be described in terms of the $O(1)$ constant, scale-independent, fluxes of conserved quantities. Using these fluxes as governing parameters, one can design dimensional arguments leading to the scaling exponents of the second-order correlation functions, energy and scalar-variance spectra, etc. Originating from the classic Kolmogorov theory of turbulence, the dimensional arguments have been one of the main and quite successful tools of investigation of turbulence and turbulent transport.

The present interest in the problem of a passive scalar advected by a white-in-time random velocity field has been motivated by Kraichnan's suggestion [1] that while the second-order moment of the scalar difference is well described by the dimensional argument, the higher-order moments scale ''anomalously,'' i.e., $S_n=\langle[T(\mathbf{x})-T(\mathbf{x}')]^{2n}\rangle \propto|\mathbf{x}-\mathbf{x}'|^{\xi_{2n}}$ with $\xi_{2n}\neq n\xi_2$. It has been also shown [2,3] that the fourth-order moments are dominated by the zero modes leading to the anomalous scaling.

In a recent paper Vergassola [4] considered a problem of a passive magnetic field $\boldsymbol{B}$ advected by a white-in-time random velocity field obeying the following equation of motion:

$$\partial_t\boldsymbol{B}+\boldsymbol{v}\cdot\nabla\boldsymbol{B}=\boldsymbol{B}\cdot\nabla\boldsymbol{v}+\nu\nabla^2\boldsymbol{B}+[\nabla\times\boldsymbol{f}]. \tag{1}$$

The vector potential $\boldsymbol{A}(\mathbf{x},t)$, defined as

$$\boldsymbol{B}(\mathbf{x},t)=[\nabla\times\boldsymbol{A}(\boldsymbol{x},t)] \tag{2}$$

satisfies the equation

$$\partial_t\boldsymbol{A}=[\boldsymbol{v}\times\boldsymbol{B}]-\nabla\varphi+\nu\nabla^2\boldsymbol{A}+\boldsymbol{f}. \tag{3}$$

The field φ in (3) is chosen to fix the gauge $\nabla\cdot\boldsymbol{A}=0$. It can be easily verified from (1) and (3) that for the case of no forcing and in the inviscid limit magnetic helicity $\mu=\int d\mathbf{x}\ \Gamma(\mathbf{x},t)$, where the helicity density $\Gamma(\mathbf{x},t)=\boldsymbol{A}\cdot\boldsymbol{B}$, is conserved.

The white-in-time incompressible $(\nabla\cdot\boldsymbol{v}=0)$ random velocity field is defined by the correlation function

$$\begin{aligned}\langle v_i(\boldsymbol{x},t)v_j(\boldsymbol{x}',t')\rangle&=\delta(t-t')D_{ij}(\boldsymbol{x}-\boldsymbol{x}')\\&\equiv\delta(t-t')[D_{ij}(0)-S_{ij}(\boldsymbol{r})]\end{aligned} \tag{4}$$

with the structure function

$$S_{ij}(\mathbf{r})=Dr^{\xi}[(\xi+2)\delta_{ij}-\xi n_i n_j]+\epsilon_{ijk}n_k h(r), \tag{5}$$

where $0\leqslant\xi\leqslant2$ and $n_i=r_i/r$. In (5) we have added (in contrast with [4]) the helical term which is proportional to the antisymmetric tensor ϵ_{ijk} and a function $h(r)$ to be discussed later. The force $\boldsymbol{f}$ is Gaussian, white in time and isotropic, but it can break the parity invariance. We assume that

$$\begin{gathered}\langle f_i(\mathbf{0},t)f_i(\boldsymbol{r},t')\rangle=\delta(t-t')F(r),\\ \frac{1}{2}\epsilon_{ijk}n_k\langle f_i(\mathbf{0},t)f_j(\boldsymbol{r},t')\rangle=\delta(t-t')F_h(r),\end{gathered} \tag{6}$$

and the force acts at large scales $r\sim L$ only. This means that when $r/L\rightarrow0$ the functions $F(r)=\text{const}-B_0^2r^2+O((r/L)^4)$ and $F_h(r)=\beta_f B_0^2 rL[1+O((r/L)^2))]$, where B_0^2 sets the level of magnetic energy and $|\beta_f|\leqslant1$ sets the level of magnetic helicity. The nonzero $F_h(r)$ explicitly leads to the violation of the parity invariance of the velocity field.

We will be interested in the two-point correlation function of the magnetic field

$$\langle B_i(\mathbf{0},t)B_i(\boldsymbol{r},t)\rangle=H(r,t). \tag{7}$$

The general form of the two-point correlation function for the vector potential $\boldsymbol{A}$ is

$$\langle A_i(\mathbf{0},t)A_j(\boldsymbol{r},t)\rangle = C_1\delta_{ij} + C_2 n_i n_j + \epsilon_{ijk} n_k C_3. \tag{8}$$

It is convenient to use the trace of correlation function tensor

$$C(r,t) = 3C_1(r,t) + C_2(r,t). \tag{9}$$

The functions C_1 and C_2 can be expressed in terms of C using incompressibility of the velocity field (see [4]). It can also be shown from (2), (7), and (8) that

$$H(r,t) = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 G) \quad \text{with } G(r,t) \equiv \frac{\partial C}{\partial r}. \tag{10}$$

Expressions (2), (8) and (10) lead to

$$\langle A_i(\mathbf{0},t)B_j(\boldsymbol{r},t)\rangle = \frac{1}{2}\epsilon_{ijk} n_k G(r,t) - \delta_{ij}\left(\frac{C_3}{r} + \frac{\partial C_3}{\partial r}\right) - n_i n_j\left(\frac{C_3}{r} - \frac{\partial C_3}{\partial r}\right). \tag{11}$$

First, we discuss the parity-invariant case $h(r)=0$ considered in [4]. For the white-in-time Gaussian velocity field equations for the correlation functions $G(r,t)$ can be derived readily using the Wick theorem. The equation of motion for $G(r,t)$ has the form

$$\frac{\partial G}{\partial t} = \frac{2}{r^2}\frac{\partial}{\partial r}\left[(\nu r^2 + D r^{2+\xi})\frac{\partial G}{\partial r}\right] + \frac{2G}{r^2}[D r^{\xi}(\xi^2 + 2\xi - 2) - 2\nu] + \frac{\partial F}{\partial r}. \tag{12}$$

Equation (12) is equivalent to the equation for $H(r,t)$ obtained in [4]. The solution of this equation has been found by Vergassola for $0<\xi\leqslant 1$ [4] (only in this case a stationary state exists). Recalling the definition (10) we find the leading contribution to the solution in the range $r \ll L$:

$$H(r) \propto r^{\gamma}, \qquad \gamma = -\frac{3+\xi}{2} + \frac{3}{2}\left(1 - \frac{1}{3}\xi(\xi+2)\right)^{1/2}. \tag{13}$$

The proportionality coefficient in (13) depends on the force amplitude. The nontrivial scaling (13) follows from the existence of nontrivial zero modes of the advection operator in the equation (12).

The most interesting feature of the Vergassola solution is that the exponent γ cannot be derived using the dimensional considerations. This fact is somewhat surprising since, at first glance, the problem of the passive magnetic field (1) is similar to that of the passive scalar with the two-point correlation function characterized by a perfectly normal scaling. Indeed, simple power counting applied to (1) easily yields an incorrect passive scalarlike exponent. The main difference between the equation for a passive scalar and (1) is the fact that (1) does not conserve the variance of magnetic field $\boldsymbol{B}^2$. The goal of this work is to elucidate the role played by the conservation laws, which do exist in the magnetic field problem, in the scaling properties of different correlation function.

The magnetic helicity is an inviscid invariant of the system governed by (1) when $f=0$. It is convenient to characterize the fluctuations of magnetic helicity in terms of the trace of this correlation function (11)

$$\Xi(r,t) \equiv \langle A_i(\mathbf{0},t)B_i(\boldsymbol{r},t)\rangle = -\frac{2}{r^2}\frac{\partial}{\partial r}(r^2 C_3). \tag{14}$$

For the white-in-time Gaussian velocity field the equations for the correlation function $\Xi(r,t)$ have the form

$$\frac{\partial \Xi}{\partial t} = \frac{2}{r^2}\frac{\partial}{\partial r}\left[(\nu r^2 + D r^{2+\xi})\frac{\partial \Xi}{\partial r}\right] - \frac{2}{r^2}\frac{\partial}{\partial r}(r^2 F_h). \tag{15}$$

It is evident from (15) that in this case the magnetic helicity correlation function obeys the equation similar to that governing the dynamics of a passive scalar advected by the velocity field $\mathbf{v}$ defined by (4). Since $F_h \neq 0$, we find the steady state scaling solution for magnetic helicity correlation function with the scaling exponent identical to that of a scalar advected by a white-in-time velocity field (see for example [2])

$$\Xi(r) - \Xi(0) = \frac{\beta_f B_0^2 L}{D(2-\xi)} r^{2-\xi}. \tag{16}$$

It is interesting that (16) holds independently of the dynamics of $H(r)$. It has been shown in [4] that the equation for $H(r)$ has a growing in time solutions when $\xi>1$. On the other hand, even for $\xi>1$ the correlation function $\Xi(r)$ is given by a steady-state solution (16) corresponding to the normal scaling. There is no contradiction since the growing in time $H(r,t)$ and the time-independent $\Xi(r)$ can exist simultaneously. The increase in the magnitude of the magnetic field vector $\boldsymbol{B}$ is accompanied by the modification of its orientation keeping magnetic helicity constant. Locally vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ should be nearly orthogonal implying that a magnetic field is locally nearly two dimensional.

Thus, we have shown that the problem of a passively advected magnetic field allows coexistence of both normal and anomalous scaling regimes for conserved and not conserved quantities, respectively. As in the Navier-Stokes turbulence and in the problem of a passive scalar advection the normal scaling is related to constant fluxes of inviscid invariants which dominate the inertial range dynamics. The dynamics of magnetic field, however, governed by (1) cannot be characterized by the constant flux. As we see, the first contribution to the right side of (1) serves as a source of magnetic field acting in the entire interval of wavenumbers $1/L<k<k_d$ where k_d is the dissipation wavenumber.

In the case of nonzero kinetic helicity $h(r)$ the well known α effect takes place [5]. Similarly to (14) the kinetic helicity correlation function is defined:

$$\langle v_i(\mathbf{0},t)\omega_i(\boldsymbol{r},t')\rangle = -\frac{2}{r^2}\frac{\partial}{\partial r}[r^2 h(r)]\delta(t-t') \tag{17}$$

where $\boldsymbol{\omega}$ is vorticity. Equations (12) and (15) take the form

$$\frac{\partial G}{\partial t}=\frac{2}{r^2}\frac{\partial}{\partial r}\left[(\nu r^2+Dr^{2+\xi})\frac{\partial G}{\partial r}\right]+\frac{2G}{r^2}[Dr^{\xi}(\xi^2+2\xi-2)-2\nu]+\frac{\partial F}{\partial r}+2\alpha(r)\frac{\partial \Xi}{\partial r} \tag{18}$$

and

$$\frac{\partial \Xi}{\partial t}=\frac{2}{r^2}\frac{\partial}{\partial r}\left[(\nu r^2+Dr^{2+\xi})\frac{\partial \Xi}{\partial r}\right]-\frac{2}{r^2}\frac{\partial}{\partial r}(r^2F_h)-\frac{2}{r^2}\frac{\partial}{\partial r}[\alpha(r)r^2G] \tag{19}$$

where

$$\alpha(r)=\frac{h(r)}{r}-\alpha_0 \quad \text{with} \quad \alpha_0=\lim_{r\to 0}\frac{h(r)}{r}. \tag{20}$$

The constant α_0 is proportional to the mean kinetic helicity (α effect). The appearance of α_0 terms in Eqs. (18) and (19) reflects the conservation law of magnetic helicity. The magnetic helicity is conserved in an inviscid case even when kinetic helicity is present.

Now we can see that in this case Eqs. (18) and (19) are coupled. It is convenient to introduce the following transformation (see [4]):

$$\psi(r,t)=\frac{rG(r,t)}{m(r)^{1/2}} \quad \text{where} \quad m(r)=\frac{1}{2(\nu+Dr^{\xi})}. \tag{21}$$

The transformation (21) as well as Eq. (12) were suggested in the seminal work of Kazantsev [6]. Using (18) and (19) we get

$$m(r)\frac{\partial}{\partial t}\left(\psi-2m(r)^{1/2}\alpha(r)\int_0^r \Xi(\rho,t)\rho^2 d\rho\right)=\frac{\partial^2\psi}{\partial r^2}-U(r)\psi \tag{22}$$

where the potential $U(r)$ in the limit of molecular diffusivity $\nu=0$ has the form

$$U(r)=\frac{2-\frac{3}{2}\xi-\frac{3}{4}\xi^2}{r^2}-\frac{\alpha^2(r)}{D^2r^{2\xi}}. \tag{23}$$

Thus, we have obtained the Schrödinger-like equation similar to that derived by Vergassola [4]. This equation is not closed. But, being interested in the properties of the steady-state solution, we set the left side of (22) to zero. Now we are looking for the zero-energy ($E=0$) solution of the Schrödinger equation with the kinetic helicity-dependent potential. The existence of the negative eigenvalue solutions ($E<0$) could imply the absence of a stationary state in this system. To prove that no stationary solution exists we need to show that the system of equations (18), (19) has a growing eigenmode satisfying the boundary conditions. These boundary conditions are the absence of singularity at small $r\sim 1/k_d$ and the requirement that correlation functions decay at large $r \gg L$. Unfortunately, as far as we understand, the existence of bounded growing eigenmodes in Eqs. (18), (19) can be proven only numerically. Therefore, the existence of negative eigenvalue solutions in the complementary Schrödinger equation should be considered only as a conjecture of an instability of a steady-state solution of the system (18), (19).

It has already been shown in [4] that for the case $\xi>1$ the Schrödinger equation has negative energy solutions that lead to the absence of a stationary state. It is clear that additional attractive potential in Eq. (22) only increases the instability. Moreover, in the case of nonzero kinetic helicity there are no stationary states even for $\xi\leqslant 1$. To demonstrate this fact we have to make more specific assumptions about kinetic helicity correlation function $h(r)$.

Obviously, the kinetic energy and kinetic helicity should be connected. It is possible to have kinetic energy without helicity, but it is impossible to have kinetic helicity without kinetic energy. It is sufficient to consider only the case of ''maximally helical'' velocity field (it is the only case when kinematic helicity correlation function has the same scaling exponent ξ). For this state velocity correlation function in Fourier space should have the form

$$4\pi k^2\langle v_i(-\boldsymbol{k})v_j(\boldsymbol{k})\rangle=E(k)\left(\delta_{ij}-\frac{k_ik_j}{k^2}+i\beta\epsilon_{ijl}\frac{k_l}{k}\right). \tag{24}$$

In (24) $|\beta|\leqslant 1$ is helicity level and ''maximal helicity'' is reached at $\beta=\pm 1$. To reproduce the scaling relation (5) we need to assume $E(k)=E_0/k^{1+\xi}$ Then it can be shown that

$$h(r)\propto\beta D\tan\left(\frac{\pi\xi}{2}\right)r^{\xi}-\beta D\frac{2\xi(\xi+2)}{3\pi(1-\xi)}\Gamma(\xi)\sin\left(\frac{\pi\xi}{2}\right)rL^{\xi-1} \tag{25}$$

where $\Gamma(\xi)$ is the gamma function, L is the infrared cutoff and $E_0\propto D$. In the case $\xi=1$ the relation (25) reduces to $h(r)\propto\beta Dr\ln(L/r)$. We also assume that (25) is valid only when $r>r_d$ and it has a dissipative cutoff at dissipation scale r_d. For $\xi\leqslant 1$ the function $\alpha(r)$ has the form

$$\alpha(r)\propto\beta D(r_d^{\xi-1}-r^{\xi-1}). \tag{26}$$

In (26) the large scale dynamics is dominated by the first term. Thus, the potential $U(r)$ in (23) is repulsive as $1/r^2$ at small r and attractive as $1/r^{2\xi}$ at the large ones. In this case the Schrödinger equation always has a negative energy state (see, for example, [7]) and no stationary state solution exists. Although the α effect on average does not lead to the increase of the total magnetic helicity it leads to the instability of the system. It is clear from (23) that this destabilization effect is $O(\alpha^2(r))$ as in the random α model discussed by Kraichnan in [8].

In conclusion, we would like to summarize the main results of this work. In the case of the parity invariant velocity field the scaling exponent of helicity correlation function is normal, i.e., can be obtained from the dimensional argument. This result is correct even in the case $\xi>1$ where the magnetic field $\boldsymbol{B}$ grows in time while the helicity correlation function is in a statistically steady state. At the same time, the dominant contribution to the scaling of the magnetic field correlation function, which is derived from a homogeneous equation, is dominated by the zero modes and is anomalous. It cannot be derived on dimensional grounds. It is interesting that even a small violation of the parity invariance of the external velocity field can make the steady state of the helicity correlation function impossible notwithstanding the conservation of total helicity in the system. The fact that the two-point correlation functions of conserved and nonconserved properties have normal and anomalous scalings, respectively, emphasizes the role of conservation laws in turbulence theory. At the present time we do not know how general this result is.

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